

Weak Approximation of G -Expectations

Yan Dolinsky* Marcel Nutz† H. Mete Soner‡

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Abstract

We introduce a notion of volatility uncertainty in discrete time and define the corresponding analogue of Peng's G -expectation. In the continuous-time limit, the resulting sublinear expectation converges weakly to the G -expectation. This can be seen as a Donsker-type result for the G -Brownian motion.

Keywords G -expectation, volatility uncertainty, weak limit theorem

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1 Introduction

The so-called G -expectation [12, 13, 14] is a nonlinear expectation advancing the notions of backward stochastic differential equations (BSDEs) [10] and g -expectations [11]; see also [2, 16] for a related theory of second order BSDEs. A G -expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear function which maps random variables ξ on the canonical space $\Omega = C([0, T]; \mathbb{R})$ to the real numbers. The symbol G refers to a given function $G : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$G(\gamma) = \frac{1}{2}(R\gamma^+ - r\gamma^-) = \frac{1}{2} \sup_{a \in [r, R]} a\gamma,$$

where $0 \leq r \leq R < \infty$ are fixed numbers. More generally, the interval $[r, R]$ is replaced by a set \mathbf{D} of nonnegative matrices in the multivariate case. The extension to a random set \mathbf{D} is studied in [9].

The construction of $\mathcal{E}^G(\xi)$ runs as follows. When $\xi = f(B_T)$, where B_T is the canonical process at time T and f is a sufficiently regular function, then $\mathcal{E}^G(\xi)$ is defined to be the initial value $u(0, 0)$ of the solution of the nonlinear backward heat equation $-\partial_t u - G(u_{xx}) = 0$ with terminal condition

*ETH Zurich, Dept. of Mathematics, CH-8092 Zurich, yan.dolinsky@math.ethz.ch

†ETH Zurich, Dept. of Mathematics, CH-8092 Zurich, marcel.nutz@math.ethz.ch

‡ETH Zurich, Dept. of Mathematics, CH-8092 Zurich, and Swiss Finance Institute, mete.soner@math.ethz.ch

$u(\cdot, T) = f$. The mapping \mathcal{E}^G can be extended to random variables of the form $\xi = f(B_{t_1}, \dots, B_{t_n})$ by a stepwise evaluation of the PDE and then to the completion \mathbb{L}_G^1 of the space of all such random variables. The space \mathbb{L}_G^1 consists of so-called quasi-continuous functions and contains in particular all bounded continuous functions on Ω ; however, not all bounded measurable functions are included (cf. [3]). While this setting is not based on a single probability measure, the so-called G -Brownian motion is given by the canonical process B “seen” under \mathcal{E}^G (cf. [14]). It reduces to the standard Brownian motion if $r = R = 1$ since \mathcal{E}^G is then the (linear) expectation under the Wiener measure.

In this note we introduce a discrete-time analogue of the G -expectation and we prove a convergence result which resembles Donsker’s theorem for the standard Brownian motion; the main purpose is to provide additional intuition for G -Brownian motion and volatility uncertainty. Our starting point is the dual view on G -expectation via volatility uncertainty [3, 4]: We consider the representation

$$\mathcal{E}^G(\xi) = \sup_{P \in \mathcal{P}} E^P[\xi], \quad (1.1)$$

where \mathcal{P} is a set of probabilities on Ω such that under any $P \in \mathcal{P}$, the canonical process B is a martingale with volatility $d\langle B \rangle/dt$ taking values in $\mathbf{D} = [r, R]$, $P \times dt$ -a.e. Therefore, \mathbf{D} can be understood as the domain of (Knightian) volatility uncertainty and \mathcal{E}^G as the corresponding worst-case expectation. In discrete-time, we translate this to uncertainty about the conditional variance of the increments. Thus we define a sublinear expectation \mathcal{E}^n on the n -step canonical space in the spirit of (1.1), replacing \mathcal{P} by a suitable set of martingale laws. A natural push-forward then yields a sublinear expectation on Ω , which we show to converge weakly to \mathcal{E}^G as $n \rightarrow \infty$, if the domain \mathbf{D} of uncertainty is scaled by $1/n$ (cf. Theorem 2.2). The proof relies on (linear) probability theory; in particular, it does not use the central limit theorem for sublinear expectations [14, 15]. The relation to the latter is nontrivial since our discrete-time models do not have independent increments. We remark that quite different approximations of the G -expectation (for the scalar case) can be found in discrete models for financial markets with transaction costs [8] or illiquidity [5].

The detailed setup and the main result are stated in Section 2, whereas the proofs and some ramifications are given in Section 3.

2 Main Result

We fix the dimension $d \in \mathbb{N}$ and denote by $|\cdot|$ the Euclidean norm on \mathbb{R}^d . Moreover, we denote by \mathbb{S}^d the space of $d \times d$ symmetric matrices and by \mathbb{S}_+^d its subset of nonnegative definite matrices. We fix a nonempty, convex and

compact set $\mathbf{D} \subseteq \mathbb{S}_+^d$; the elements of \mathbf{D} will be the possible values of our volatility processes.

Continuous-Time Formulation. Let $\Omega = C([0, T]; \mathbb{R}^d)$ be the space of d -dimensional continuous paths $\omega = (\omega_t)_{0 \leq t \leq T}$ with time horizon $T \in (0, \infty)$, endowed with the uniform norm $\|\omega\|_\infty = \sup_{0 \leq t \leq T} |\omega_t|$. We denote by $B = (B_t)_{0 \leq t \leq T}$ the canonical process $B_t(\omega) = \omega_t$ and by $\mathcal{F}_t := \sigma(B_s, 0 \leq s \leq t)$ the canonical filtration. A probability measure P on Ω is called a *martingale law* if B is a P -martingale and $B_0 = 0$ P -a.s. (All our martingales will start at the origin.) We set

$$\mathcal{P}_{\mathbf{D}} = \{P \text{ martingale law on } \Omega : d\langle B \rangle_t/dt \in \mathbf{D}, P \times dt\text{-a.e.}\},$$

where $\langle B \rangle$ denotes the matrix-valued process of quadratic covariations. We can then define the sublinear expectation

$$\mathcal{E}_{\mathbf{D}}(\xi) := \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi] \quad \text{for any random variable } \xi : \Omega \rightarrow \mathbb{R}$$

such that ξ is \mathcal{F}_T -measurable and $E^P|\xi| < \infty$ for all $P \in \mathcal{P}_{\mathbf{D}}$. The mapping $\mathcal{E}_{\mathbf{D}}$ coincides with the G -expectation (on its domain \mathbb{L}_G^1) if $G : \mathbb{S}^d \rightarrow \mathbb{R}$ is (half) the support function of \mathbf{D} ; i.e., $G(\Gamma) = \sup_{A \in \mathbf{D}} \text{trace}(\Gamma A)/2$. Indeed, this follows from [3] with an additional density argument as detailed in Remark 3.6 below.

Discrete-Time Formulation. Given $n \in \mathbb{N}$, we consider $(\mathbb{R}^d)^{n+1}$ as the canonical space of d -dimensional paths in discrete time $k = 0, 1, \dots, n$. We denote by $X^n = (X_k^n)_{k=0}^n$ the canonical process defined by $X_k^n(x) = x_k$ for $x = (x_0, \dots, x_n) \in (\mathbb{R}^d)^{n+1}$. Moreover, $\mathcal{F}_k^n = \sigma(X_i^n, i = 0, \dots, k)$ defines the canonical filtration $(\mathcal{F}_k^n)_{k=0}^n$. We also introduce $0 \leq r_{\mathbf{D}} \leq R_{\mathbf{D}} < \infty$ such that $[r_{\mathbf{D}}, R_{\mathbf{D}}]$ is the spectrum of \mathbf{D} ; i.e.,

$$r_{\mathbf{D}} = \inf_{\Gamma \in \mathbf{D}} \|\Gamma^{-1}\|^{-1} \quad \text{and} \quad R_{\mathbf{D}} = \sup_{\Gamma \in \mathbf{D}} \|\Gamma\|,$$

where $\|\cdot\|$ denotes the operator norm and we set $r_{\mathbf{D}} := 0$ if \mathbf{D} has an element which is not invertible. We note that $[r_{\mathbf{D}}, R_{\mathbf{D}}] = \mathbf{D}$ if $d = 1$. Finally, a probability measure P on $(\mathbb{R}^d)^{n+1}$ is called a martingale law if X^n is a P -martingale and $X_0^n = 0$ P -a.s. Denoting by $\Delta X_k^n = X_k^n - X_{k-1}^n$ the increments of X^n , we can now set

$$\mathcal{P}_{\mathbf{D}}^n = \left\{ P \text{ martingale law on } (\mathbb{R}^d)^{n+1} : \begin{array}{l} \text{for } k = 1, \dots, n, \\ E^P[\Delta X_k^n (\Delta X_k^n)' | \mathcal{F}_{k-1}^n] \in \mathbf{D} \text{ and } d^2 r_{\mathbf{D}} \leq |\Delta X_k^n|^2 \leq d^2 R_{\mathbf{D}}, P\text{-a.s.} \end{array} \right\},$$

where prime ($'$) denotes transposition. Note that ΔX_k^n is a column vector, so that $\Delta X^n (\Delta X^n)'$ takes values in \mathbb{S}_+^d . We introduce the sublinear expectation

$$\mathcal{E}_{\mathbf{D}}^n(\psi) := \sup_{P \in \mathcal{P}_{\mathbf{D}}^n} E^P[\psi] \quad \text{for any random variable } \psi : (\mathbb{R}^d)^{n+1} \rightarrow \mathbb{R}$$

such that ψ is \mathcal{F}_n^n -measurable and $E^P|\psi| < \infty$ for all $P \in \mathcal{P}_{\mathbf{D}}^n$, and we think of $\mathcal{E}_{\mathbf{D}}^n$ as a discrete-time analogue of the G -expectation.

Remark 2.1. The second condition in the definition of $\mathcal{P}_{\mathbf{D}}^n$ is motivated by the desire to generate the volatility uncertainty by a *small* set of scenarios; we remark that the main results remain true if, e.g., the lower bound $r_{\mathbf{D}}$ is omitted and the upper bound $R_{\mathbf{D}}$ replaced by any other condition yielding tightness. Our bounds are chosen so that

$$\mathcal{P}_{\mathbf{D}}^n = \{P \text{ martingale law on } (\mathbb{R}^d)^{n+1} : \Delta X^n(\Delta X^n)' \in \mathbf{D}, P\text{-a.s.}\} \quad \text{if } d = 1.$$

Continuous-Time Limit. To compare our objects from the two formulations, we shall extend any discrete path $x \in (\mathbb{R}^d)^{n+1}$ to a continuous path $\hat{x} \in \Omega$ by linear interpolation. More precisely, we define the interpolation operator

$$\begin{aligned} \hat{\cdot}: (\mathbb{R}^d)^{n+1} &\rightarrow \Omega, \quad x = (x_0, \dots, x_n) \mapsto \hat{x} = (\hat{x}_t)_{0 \leq t \leq T}, \quad \text{where} \\ \hat{x}_t &:= ([nt/T] + 1 - nt/T)x_{[nt/T]} + (nt/T - [nt/T])x_{[nt/T]+1} \end{aligned}$$

and $[y] := \max\{m \in \mathbb{Z} : m \leq y\}$ for $y \in \mathbb{R}$. In particular, if X^n is the canonical process on $(\mathbb{R}^d)^{n+1}$ and ξ is a random variable on Ω , then $\xi(\hat{X}^n)$ defines a random variable on $(\mathbb{R}^d)^{n+1}$. This allows us to define the following push-forward of $\mathcal{E}_{\mathbf{D}}^n$ to a continuous-time object,

$$\hat{\mathcal{E}}_{\mathbf{D}}^n(\xi) := \mathcal{E}_{\mathbf{D}}^n(\xi(\hat{X}^n)) \quad \text{for } \xi : \Omega \rightarrow \mathbb{R}$$

being suitably integrable.

Our main result states that this sublinear expectation with discrete-time volatility uncertainty converges to the G -expectation as the number n of periods tends to infinity, if the domain of volatility uncertainty is scaled as $\mathbf{D}/n := \{n^{-1}\Gamma : \Gamma \in \mathbf{D}\}$.

Theorem 2.2. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq c(1 + \|\omega\|_{\infty})^p$ for some constants $c, p > 0$. Then $\hat{\mathcal{E}}_{\mathbf{D}/n}^n(\xi) \rightarrow \mathcal{E}_{\mathbf{D}}(\xi)$ as $n \rightarrow \infty$; that is,*

$$\sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\hat{X}^n)] \rightarrow \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]. \quad (2.1)$$

We shall see that all expressions in (2.1) are well defined and finite. Moreover, we will show in Theorem 3.8 that the result also holds true for a “strong” formulation of volatility uncertainty.

Remark 2.3. Theorem 2.2 cannot be extended to the case where ξ is merely in \mathbb{L}_G^1 , which is defined as the completion of $C_b(\Omega; \mathbb{R})$ under the norm $\|\xi\|_{\mathbb{L}_G^1} := \sup\{E^P|\xi|, P \in \mathcal{P}_{\mathbf{D}}\}$. This is because $\|\cdot\|_{\mathbb{L}_G^1}$ “does not see” the discrete-time objects, as illustrated by the following example. Assume

for simplicity that $0 \notin \mathbf{D}$ and let $A \subset \Omega$ be the set of paths with finite variation. Since $P(A) = 0$ for any $P \in \mathcal{P}_{\mathbf{D}}$, we have $\xi := 1 - \mathbf{1}_A = 1$ in \mathbb{L}_G^1 and the right hand side of (2.1) equals one. However, the trajectories of \widehat{X}^n lie in A , so that $\xi(\widehat{X}^n) \equiv 0$ and the left hand side of (2.1) equals zero.

In view of the previous remark, we introduce a smaller space \mathbb{L}_*^1 , defined as the completion of $C_b(\Omega; \mathbb{R})$ under the norm

$$\|\xi\|_* := \sup_{Q \in \mathcal{Q}} E^Q |\xi|, \quad \mathcal{Q} := \mathcal{P}_{\mathbf{D}} \cup \{P \circ (\widehat{X}^n)^{-1} : P \in \mathcal{P}_{\mathbf{D}/n}^n, n \in \mathbb{N}\}. \quad (2.2)$$

If ξ is as in Theorem 2.2, then $\xi \in \mathbb{L}_*^1$ by Lemma 3.4 below and so the following is a generalization of Theorem 2.2.

Corollary 2.4. *Let $\xi \in \mathbb{L}_*^1$. Then $\widehat{\mathcal{E}}_{\mathbf{D}/n}^n(\xi) \rightarrow \mathcal{E}_{\mathbf{D}}(\xi)$ as $n \rightarrow \infty$.*

Proof. This follows from Theorem 2.2 by approximation, using that $\|\xi\|_*$ and $\sup\{E^P |\xi| : P \in \mathcal{P}_{\mathbf{D}}\} + \sup\{E^P |\xi(\widehat{X}^n)| : P \in \mathcal{P}_{\mathbf{D}/n}^n, n \in \mathbb{N}\}$ are equivalent norms. \square

3 Proofs and Ramifications

In the next two subsections, we prove separately two inequalities that jointly imply Theorem 2.2 and a slightly stronger result, reported in Theorem 3.8.

3.1 First Inequality

In this subsection we prove the first inequality of (2.1), namely that

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P [\xi(\widehat{X}^n)] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P [\xi]. \quad (3.1)$$

The essential step in this proof is a stability result for the volatility (see Lemma 3.3(ii) below); the necessary tightness follows from the compactness of \mathbf{D} ; i.e., from $R_{\mathbf{D}} < \infty$. We shall denote $\lambda \mathbf{D} = \{\lambda \Gamma : \Gamma \in \mathbf{D}\}$ for $\lambda \in \mathbb{R}$.

Lemma 3.1. *Given $p \in [1, \infty)$, there exists a universal constant $K > 0$ such that for all $0 \leq k \leq l \leq n$ and $P \in \mathcal{P}_{\mathbf{D}}^n$,*

- (i) $E^P [\sup_{k=0, \dots, n} |X_k^n|^{2p}] \leq K(nR_{\mathbf{D}})^p$,
- (ii) $E^P |X_l^n - X_k^n|^4 \leq KR_{\mathbf{D}}^2 (l - k)^2$,
- (iii) $E^P [(X_l^n - X_k^n)(X_l^n - X_k^n)' | \mathcal{F}_k^n] \in (l - k)\mathbf{D}$ *P*-a.s.

Proof. We set $X := X^n$ to ease the notation.

(i) Let $p \in [1, \infty)$. By the Burkholder-Davis-Gundy (BDG) inequalities there exists a universal constant $C = C(p, d)$ such that

$$E^P \left[\sup_{k=0, \dots, n} |X_k^n|^{2p} \right] \leq CE^P \|[X]_n\|^p.$$

In view of $P \in \mathcal{P}_{\mathbf{D}}^n$, we have $\|[X]_n\| = \|\sum_{i=1}^n \Delta X_i (\Delta X_i)'\| \leq nd^2 R_{\mathbf{D}}$ P -a.s.

(ii) The BDG inequalities yield a universal constant C such that

$$E^P |X_l - X_k|^4 \leq C E^P \|[X]_l - [X]_k\|^2.$$

Similarly as in (i), $P \in \mathcal{P}_{\mathbf{D}}^n$ implies that $\|[X]_l - [X]_k\| \leq (l - k)d^2 R_{\mathbf{D}}$ P -a.s.

(iii) The orthogonality of the martingale increments yields that

$$E^P[(X_l - X_k)(X_l - X_k)' | \mathcal{F}_k^n] = \sum_{i=k+1}^l E^P[\Delta X_i (\Delta X_i)' | \mathcal{F}_k^n].$$

Since $E^P[\Delta X_i (\Delta X_i)' | \mathcal{F}_{i-1}^n] \in \mathbf{D}$ P -a.s. and since \mathbf{D} is convex,

$$E^P[\Delta X_i (\Delta X_i)' | \mathcal{F}_k^n] = E^P[E^P[\Delta X_i (\Delta X_i)' | \mathcal{F}_{i-1}^n] | \mathcal{F}_k^n]$$

again takes values in \mathbf{D} . It remains to observe that if $\Gamma_1, \dots, \Gamma_m \in \mathbf{D}$, then $\Gamma_1 + \dots + \Gamma_m \in m\mathbf{D}$ by convexity. \square

The following lemma shows in particular that all expressions in Theorem 2.2 are well defined and finite.

Lemma 3.2. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be as in Theorem 2.2. Then $\|\xi\|_* < \infty$; that is,*

$$\sup_{n \in \mathbb{N}} \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P |\xi(\widehat{X}^n)| < \infty \quad \text{and} \quad \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P |\xi| < \infty. \quad (3.2)$$

Proof. Let $n \in \mathbb{N}$ and $P \in \mathcal{P}_{\mathbf{D}/n}^n$. By the assumption on ξ , there exist constants $c, p > 0$ such that

$$E^P |\xi(\widehat{X}^n)| \leq c + c E^P \left[\sup_{0 \leq t \leq T} |\widehat{X}_t^n|^p \right] \leq c + c E^P \left[\sup_{k=0, \dots, n} |X_k^n|^p \right].$$

Hence Lemma 3.1(i) and the observation that $R_{\mathbf{D}/n} = R_{\mathbf{D}}/n$ yield that $E^P |\xi(\widehat{X}^n)| \leq K R_{\mathbf{D}}^{p/2}$ and the first claim follows. The second claim similarly follows from the estimate that $E^P [\sup_{0 \leq t \leq T} |B_t|^p] \leq C_p$ for all $P \in \mathcal{P}_{\mathbf{D}}$, which is obtained from the BDG inequalities by using that \mathbf{D} is bounded. \square

We can now prove the key result of this subsection.

Lemma 3.3. *For each $n \in \mathbb{N}$, let $\{M^n = (M_k^n)_{k=0}^n, \tilde{P}^n\}$ be a martingale with law $P^n \in \mathcal{P}_{\mathbf{D}/n}^n$ on $(\mathbb{R}^d)^{n+1}$ and let Q^n be the law of \widehat{M}^n on Ω . Then*

- (i) *the sequence (Q^n) is tight on Ω ,*
- (ii) *any cluster point of (Q^n) is an element of $\mathcal{P}_{\mathbf{D}}$.*

Proof. (i) Let $0 \leq s \leq t \leq T$. As $R_{\mathbf{D}/n} = R_{\mathbf{D}}/n$, Lemma 3.1(ii) implies that

$$E^{Q^n} |B_t - B_s|^4 = E^{\tilde{P}^n} |\widehat{M}_t^n - \widehat{M}_s^n|^4 \leq C|t - s|^2$$

for a constant $C > 0$. Hence (Q^n) is tight by the moment criterion.

(ii) Let Q be a cluster point, then B is a Q -martingale as a consequence of the uniform integrability implied by Lemma 3.1(i) and it remains to show that $d\langle B \rangle_t/dt \in \mathbf{D}$ holds $Q \times dt$ -a.e. It will be useful to characterize \mathbf{D} by scalar inequalities: given $\Gamma \in \mathbb{S}^d$, the separating hyperplane theorem implies that

$$\Gamma \in \mathbf{D} \quad \text{if and only if} \quad \ell(\Gamma) \leq C_{\mathbf{D}}^\ell := \sup_{A \in \mathbf{D}} \ell(A) \quad \text{for all} \quad \ell \in (\mathbb{S}^d)^*, \quad (3.3)$$

where $(\mathbb{S}^d)^*$ is the set of all linear functionals $\ell : \mathbb{S}^d \rightarrow \mathbb{R}$.

Let $H : [0, T] \times \Omega \rightarrow [0, 1]$ be a continuous and adapted function and let $\ell \in (\mathbb{S}^d)^*$. We fix $0 \leq s < t \leq T$ and denote $\Delta_{s,t}Y := Y_t - Y_s$ for a process $Y = (Y_u)_{0 \leq u \leq T}$. Let $\varepsilon > 0$ and let $\tilde{\mathbf{D}}$ be any neighborhood of \mathbf{D} , then for n sufficiently large,

$$E^{\tilde{P}^n} \left[(\Delta_{s,t} \widehat{M}^n) (\Delta_{s,t} \widehat{M}^n)' \middle| \sigma(\widehat{M}_u^n, 0 \leq u \leq s - \varepsilon) \right] \in (t - s) \tilde{\mathbf{D}} \quad \tilde{P}^n\text{-a.s.}$$

as a consequence of Lemma 3.1(iii). Since $\tilde{\mathbf{D}}$ was arbitrary, it follows by (3.3) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} E^{Q^n} [H(s - \varepsilon, B) \{ \ell((\Delta_{s,t} B)(\Delta_{s,t} B)') - C_{\mathbf{D}}^\ell(t - s) \}] \\ &= \limsup_{n \rightarrow \infty} E^{\tilde{P}^n} [H(s - \varepsilon, \widehat{M}^n) \{ \ell((\Delta_{s,t} \widehat{M}^n)(\Delta_{s,t} \widehat{M}^n)') - C_{\mathbf{D}}^\ell(t - s) \}] \leq 0. \end{aligned}$$

Using (3.2) with $\xi(\omega) = \|\omega\|_\infty^2$, we may pass to the limit and conclude that

$$E^Q [H(s - \varepsilon, B) \ell((\Delta_{s,t} B)(\Delta_{s,t} B)')] \leq E^Q [H(s - \varepsilon, B) C_{\mathbf{D}}^\ell(t - s)]. \quad (3.4)$$

Since $H(s - \varepsilon, B)$ is \mathcal{F}_s -measurable and

$$E^Q [(\Delta_{s,t} B)(\Delta_{s,t} B)' | \mathcal{F}_s] = E^Q [B_t B_t' - B_s B_s' | \mathcal{F}_s] = E^Q [\langle B \rangle_t - \langle B \rangle_s | \mathcal{F}_s]$$

as B is a square-integrable Q -martingale, (3.4) is equivalent to

$$E^Q [H(s - \varepsilon, B) \ell(\langle B \rangle_t - \langle B \rangle_s)] \leq E^Q [H(s - \varepsilon, B) C_{\mathbf{D}}^\ell(t - s)].$$

Using the continuity of H and dominated convergence as $\varepsilon \rightarrow 0$, we obtain

$$E^Q [H(s, B) \ell(\langle B \rangle_t - \langle B \rangle_s)] \leq E^Q [H(s, B) C_{\mathbf{D}}^\ell(t - s)]$$

and then it follows that

$$E^Q \left[\int_0^T H(t, B) \ell(d\langle B \rangle_t) \right] \leq E^Q \left[\int_0^T H(t, B) C_{\mathbf{D}}^\ell dt \right].$$

By an approximation argument, this inequality extends to functions H which are measurable instead of continuous. It follows that $\ell(d\langle B \rangle_t/dt) \leq C_{\mathbf{D}}^\ell$ holds $Q \times dt$ -a.e., and since $\ell \in (\mathbb{S}^d)^*$ was arbitrary, (3.3) shows that $d\langle B \rangle_t/dt \in \mathbf{D}$ holds $Q \times dt$ -a.e. \square

We can now deduce the first inequality of Theorem 2.2 as follows.

Proof of (3.1). Let ξ be as in Theorem 2.2 and let $\varepsilon > 0$. For each $n \in \mathbb{N}$ there exists an ε -optimizer $P^n \in \mathcal{P}_{\mathbf{D}/n}^n$; i.e., if Q^n denotes the law of \widehat{X}^n on Ω under P_n , then

$$E^{Q^n}[\xi] = E^{P^n}[\xi(\widehat{X}^n)] \geq \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] - \varepsilon.$$

By Lemma 3.3, the sequence (Q^n) is tight and any cluster point belongs to $\mathcal{P}_{\mathbf{D}}$. Since ξ is continuous and (3.2) implies $\sup_n E^{Q^n}[\xi] < \infty$, tightness yields that $\limsup_n E^{Q^n}[\xi] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]$. Therefore,

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi] + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that (3.1) holds. \square

Finally, we also prove the statement preceding Corollary 2.4.

Lemma 3.4. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be as in Theorem 2.2. Then $\xi \in \mathbb{L}_*^1$.*

Proof. We show that $\xi^m := (\xi \wedge m) \vee m$ converges to ξ in the norm $\|\cdot\|_*$ as $m \rightarrow \infty$, or equivalently, that the upper expectation $\sup\{E^Q[\cdot] : Q \in \mathcal{Q}\}$ is continuous along the decreasing sequence $|\xi - \xi^m|$, where \mathcal{Q} is as in (2.2). Indeed, \mathcal{Q} is tight by (the proof of) Lemma 3.3. Using that $\|\xi\|_* < \infty$ by Lemma 3.2, we can then argue as in the proof of [3, Theorem 12] to obtain the claim. \square

3.2 Second Inequality

The main purpose of this subsection is to show the second inequality “ \geq ” of (2.1). Our proof will yield a more precise version of Theorem 2.2. Namely, we will include “strong” formulations of volatility uncertainty both in discrete and in continuous time; i.e., consider laws generated by integrals with respect to a fixed random walk (resp. Brownian motion). In the financial interpretation, this means that the uncertainty can be generated by *complete* market models.

Strong Formulation in Continuous Time. Here we shall consider *Brownian* martingales: with P_0 denoting the Wiener measure, we define

$$\mathcal{Q}_{\mathbf{D}} = \left\{ P_0 \circ \left(\int f(t, B) dB_t \right)^{-1} : f \in C([0, T] \times \Omega; \sqrt{\mathbf{D}}) \text{ adapted} \right\},$$

where $\sqrt{\mathbf{D}} = \{\sqrt{\Gamma} : \Gamma \in \mathbf{D}\}$. (For $\Gamma \in \mathbb{S}_+^d$, $\sqrt{\Gamma}$ denotes the unique square-root in \mathbb{S}_+^d .) We note that $\mathcal{Q}_{\mathbf{D}}$ is a (typically strict) subset of $\mathcal{P}_{\mathbf{D}}$. The elements of $\mathcal{Q}_{\mathbf{D}}$ with nondegenerate f have the predictable representation property; i.e., they correspond to a complete market in the terminology of mathematical finance. We have the following density result; the proof is deferred to the end of the section.

Proposition 3.5. *The convex hull of $\mathcal{Q}_{\mathbf{D}}$ is a weakly dense subset of $\mathcal{P}_{\mathbf{D}}$.*

We can now deduce the connection between $\mathcal{E}_{\mathbf{D}}$ and the G -expectation associated with \mathbf{D} .

Remark 3.6. (i) Proposition 3.5 implies that

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} E^P[\xi] = \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi], \quad \xi \in C_b(\Omega; \mathbb{R}). \quad (3.5)$$

In [3, Section 3] it is shown that the G -expectation as introduced in [12, 13] coincides with the mapping $\xi \mapsto \sup_{P \in \mathcal{Q}_{\mathbf{D}}^*} E^P[\xi]$ for a certain set $\mathcal{Q}_{\mathbf{D}}^*$ satisfying $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{Q}_{\mathbf{D}}^* \subseteq \mathcal{P}_{\mathbf{D}}$. In particular, we deduce that the right hand side of (3.5) is indeed equal to the G -expectation, as claimed in Section 2.

(ii) A result similar to Proposition 3.5 can also be deduced from [17, Proposition 3.4.], which relies on a PDE-based verification argument of stochastic control. We include a (possibly more enlightening) probabilistic proof at the end of the section.

Strong Formulation in Discrete Time. For fixed $n \in \mathbb{N}$, we consider

$$\Omega_n := \{\omega = (\omega_1, \dots, \omega_n) : \omega_i \in \{1, \dots, d+1\}, i = 1, \dots, n\}$$

equipped with its power set and let $P_n := \{(d+1)^{-1}, \dots, (d+1)^{-1}\}^n$ be the product probability associated with the uniform distribution. Moreover, let ξ_1, \dots, ξ_n be an i.i.d. sequence of \mathbb{R}^d -valued random variables on Ω_n such that $|\xi_k| = d$ and such that the components of ξ_k are orthonormal in $L^2(P_n)$, for each $k = 1, \dots, n$. Let $Z_k = \sum_{l=1}^k \xi_l$ be the associated random walk. Then, we consider martingales M^f which are discrete-time integrals of Z of the form

$$M_k^f = \sum_{l=1}^k f(l-1, Z) \Delta Z_l,$$

where f is measurable and adapted with respect to the filtration generated by Z ; i.e., $f(l, Z)$ depends only on $Z|_{\{0, \dots, l\}}$. We define

$$\mathcal{Q}_{\mathbf{D}}^n = \left\{ P_n \circ (M^f)^{-1}; f : \{0, \dots, n-1\} \times (\mathbb{R}^d)^{n+1} \rightarrow \sqrt{\mathbf{D}} \text{ measurable, adapted} \right\}.$$

To see that $\mathcal{Q}_{\mathbf{D}}^n \subseteq \mathcal{P}_{\mathbf{D}}^n$, we note that $\Delta_k M^f = f(k-1, Z)\xi_k$ and the orthonormality property of ξ_k yield

$$E^{P_n} [\Delta_k M^f (\Delta_k M^f)' | \sigma(Z_1, \dots, Z_{k-1})] = f(k-1, Z)^2 \in \mathbf{D} \quad P_n\text{-a.s.},$$

while $|\xi_k| = d$ and $f^2 \in \mathbf{D}$ imply that

$$\|\Delta_k M^f (\Delta_k M^f)'\| = |f(k-1, Z)\xi_k|^2 \in [d^2 r_{\mathbf{D}}, d^2 R_{\mathbf{D}}] \quad P_n\text{-a.s.}$$

Remark 3.7. We recall from [7] that such ξ_1, \dots, ξ_n can be constructed as follows. Let A be an orthogonal $(d+1) \times (d+1)$ matrix whose last row is $((d+1)^{-1/2}, \dots, (d+1)^{-1/2})$ and let $v_l \in \mathbb{R}^d$ be column vectors such that $[v_1, \dots, v_{d+1}]$ is the matrix obtained from A by deleting the last row. Setting $\xi_k(\omega) := (d+1)^{1/2} v_{\omega_k}$ for $\omega = (\omega_1, \dots, \omega_n)$ and $k = 1, \dots, n$, the above requirements are satisfied.

We can now formulate a result which includes Theorem 2.2.

Theorem 3.8. *Let $\xi : \Omega \rightarrow \mathbb{R}$ be as in Theorem 2.2. Then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{P \in \mathcal{Q}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] &= \lim_{n \rightarrow \infty} \sup_{P \in \mathcal{P}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \\ &= \sup_{P \in \mathcal{Q}_{\mathbf{D}}} E^P[\xi] \\ &= \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi]. \end{aligned} \tag{3.6}$$

Proof. Since $\mathcal{Q}_{\mathbf{D}/n}^n \subseteq \mathcal{P}_{\mathbf{D}/n}^n$ for each $n \geq 1$, the inequality (3.1) yields that

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{Q}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \leq \sup_{P \in \mathcal{P}_{\mathbf{D}}} E^P[\xi].$$

As the equality in (3.6) follows from Proposition 3.5, it remains to show that

$$\liminf_{n \rightarrow \infty} \sup_{P \in \mathcal{Q}_{\mathbf{D}/n}^n} E^P[\xi(\widehat{X}^n)] \geq \sup_{P \in \mathcal{Q}_{\mathbf{D}}} E^P[\xi].$$

To this end, let $P \in \mathcal{Q}_{\mathbf{D}}$; i.e., P is the law of a martingale of the form

$$M = \int f(t, W) dW_t,$$

where W is a Brownian motion and $f \in C([0, T] \times \Omega; \sqrt{\mathbf{D}})$ is an adapted function. We shall construct martingales $M^{(n)}$ whose laws are in $\mathcal{Q}_{\mathbf{D}/n}^n$ and tend to P .

For $n \geq 1$, let $Z_k^{(n)} = \sum_{l=1}^k \xi_l$ be the random walk on (Ω_n, P_n) as introduced before Remark 3.7. Let

$$W_t^{(n)} := n^{-1/2} \sum_{k=1}^{[nt/T]} \xi_k, \quad 0 \leq t \leq T$$

be the piecewise constant càdlàg version of the scaled random walk and let $\hat{W}^{(n)} := n^{-1/2} \widehat{Z}^{(n)}$ be its continuous counterpart obtained by linear interpolation. It follows from the central limit theorem that

$$(W^{(n)}, \hat{W}^{(n)}) \Rightarrow (W, W) \quad \text{on} \quad D([0, T]; \mathbb{R}^{2d}),$$

the space of càdlàg paths equipped with the Skorohod topology. Moreover, since f is continuous, we also have that

$$(W^{(n)}, f([nt/T]T/n, \hat{W}^{(n)})) \Rightarrow (W, f(t, W)) \quad \text{on} \quad D([0, T]; \mathbb{R}^{d+d^2}).$$

Thus, if we introduce the discrete-time integral

$$M_k^{(n)} := \sum_{l=1}^k f((l-1)T/n, \hat{W}^{(n)}) \left(\hat{W}_{lT/n}^{(n)} - \hat{W}_{(l-1)T/n}^{(n)} \right),$$

it follows from the stability of stochastic integrals (see [6, Theorem 4.3 and Definition 4.1]) that

$$\left(M_{[nt/T]}^{(n)} \right)_{0 \leq t \leq T} \Rightarrow M \quad \text{on} \quad D([0, T]; \mathbb{R}^d).$$

Moreover, since the increments of $M^{(n)}$ uniformly tend to 0 as $n \rightarrow \infty$, it also follows that

$$\widehat{M}^{(n)} \Rightarrow M \quad \text{on} \quad \Omega.$$

As f^2/n takes values in \mathbf{D}/n , the law of $M^{(n)}$ is contained in $\mathcal{Q}_{\mathbf{D}/n}^n$ and the proof is complete. \square

It remains to give the proof of Proposition 3.5, which we will obtain by a randomization technique. Since similar arguments, at least for the scalar case, can be found elsewhere (e.g., [8, Section 5]), we shall be brief.

Proof of Proposition 3.5. We may assume without loss of generality that

$$\text{there exists an invertible element } \Gamma_* \in \mathbf{D}. \quad (3.7)$$

Indeed, using that \mathbf{D} is a convex subset of \mathbb{S}_+^d , we observe that (3.7) is equivalent to $K = \{0\}$ for $K := \bigcap_{\Gamma \in \mathbf{D}} \ker \Gamma$. If $k = \dim K > 0$, a change

of coordinates bring us to the situation where K corresponds to the last k coordinates of \mathbb{R}^d . We can then reduce all considerations to \mathbb{R}^{d-k} and thereby recover the situation of (3.7).

1. *Regularization.* We first observe that the set

$$\{P \in \mathcal{P}_{\mathbf{D}} : d\langle B \rangle_t/dt \geq \varepsilon \mathbb{1}_d P \times dt\text{-a.e. for some } \varepsilon > 0\} \quad (3.8)$$

is weakly dense in $\mathcal{P}_{\mathbf{D}}$. (Here $\mathbb{1}_d$ denotes the unit matrix.) Indeed, let M be a martingale whose law is in $\mathcal{P}_{\mathbf{D}}$. Recall (3.7) and let N be an independent continuous Gaussian martingale with $d\langle N \rangle_t/dt = \Gamma_*$. For $\lambda \uparrow 1$, the law of $\lambda M + (1 - \lambda)N$ tends to the law of M and is contained in the set (3.8), since \mathbf{D} is convex.

2. *Discretization.* Next, we reduce to martingales with piecewise constant volatility. Let M be a martingale whose law belongs to (3.8). We have

$$M = \int \sigma_t dW_t \quad \text{for } \sigma_t := \sqrt{d\langle M \rangle/dt} \quad \text{and} \quad W := \int \sigma_t^{-1} dM_t,$$

where W is a Brownian motion by Lévy's theorem. For $n \geq 1$, we introduce $M^{(n)} = \int \sigma_t^{(n)} dW_t$, where $\sigma^{(n)}$ is an \mathbb{S}_+^d -valued piecewise constant process satisfying

$$(\sigma_t^{(n)})^2 = \Pi_{\mathbf{D}} \left[\left(\frac{n}{T} \int_{(k-1)T/n}^{kT/n} \sigma_s ds \right)^2 \right], \quad t \in (kT/n, (k+1)T/n]$$

for $k = 1, \dots, n-1$, where $\Pi_{\mathbf{D}} : \mathbb{S}^d \rightarrow \mathbf{D}$ is the Euclidean projection. On $[0, T/n]$ one can take, e.g., $\sigma^{(n)} := \sqrt{\Gamma_*}$. We then have

$$E \|\langle M - M^{(n)} \rangle_T\| = E \int_0^T \|\sigma_t - \sigma_t^{(n)}\|^2 dt \rightarrow 0$$

and in particular $M^{(n)}$ converges weakly to M .

3. *Randomization.* Consider a martingale of the form $M = \int \sigma_t dW_t$, where W is a Brownian motion on some given filtered probability space and σ is an adapted $\sqrt{\mathbf{D}}$ -valued process which is piecewise constant; i.e.,

$$\sigma = \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1})} \sigma(k) \quad \text{for some } 0 = t_0 < t_1 < \dots < t_n = T$$

and some $n \geq 1$. Consider also a second probability space carrying a Brownian motion \tilde{W} and a sequence U^1, \dots, U^n of $\mathbb{R}^{d \times d}$ -valued random variables such that the components $\{U_{ij}^k : 1 \leq i, j \leq d; 1 \leq k \leq n\}$ are i.i.d. uniformly distributed on $(0, 1)$ and independent of \tilde{W} .

Using the existence of regular conditional probability distributions, we can construct functions $\Theta_k : C([0, t_k]; \mathbb{R}^d) \times (0, 1)^{d^2} \times \dots \times (0, 1)^{d^2} \rightarrow \sqrt{\mathbf{D}}$ such that the random variables $\tilde{\sigma}(k) := \Theta_k(\tilde{W}|_{[0, t_k]}, U^1, \dots, U^k)$ satisfy

$$\{\tilde{W}, \tilde{\sigma}(0), \dots, \tilde{\sigma}(n-1)\} = \{W, \sigma(0), \dots, \sigma(n-1)\} \quad \text{in law.} \quad (3.9)$$

We can then consider the volatility corresponding to a fixed realization of U^1, \dots, U^n . Indeed, for $u = (u^1, \dots, u^n) \in (0, 1)^{nd^2}$, let

$$\tilde{\sigma}(k; u) := \Theta_k(\tilde{W}|_{[0, t_k]}, u^1, \dots, u^k)$$

and consider $\tilde{M}^u = \int \tilde{\sigma}_t^u d\tilde{W}_t$, where $\tilde{\sigma}^u := \sum_{k=0}^{n-1} \mathbf{1}_{[t_k, t_{k+1})} \tilde{\sigma}(k; u)$. For any $F \in C_b(\Omega; \mathbb{R})$, the equality (3.9) and Fubini's theorem yield that

$$\begin{aligned} E[F(M)] &= E[F(\tilde{M}^{(U^1, \dots, U^n)})] = \int_{(0,1)^{nd^2}} E[F(\tilde{M}^u)] du \\ &\leq \sup_{u \in (0,1)^{nd^2}} E[F(\tilde{M}^u)]. \end{aligned}$$

Hence, by the Hahn-Banach theorem, the law of M is contained in the weak closure of the convex hull of the laws of $\{\tilde{M}^u : u \in (0, 1)^{nd^2}\}$. We note that \tilde{M}^u is of the form $\tilde{M}^u = \int g(t, \tilde{W}) d\tilde{W}_t$ with a measurable, adapted, $\sqrt{\mathbf{D}}$ -valued function g , for each fixed u .

4. *Smoothing.* As \mathcal{Q}_D is defined through continuous functions, it remains to approximate g by a continuous function f . Let $g : [0, T] \times \Omega \rightarrow \sqrt{\mathbf{D}}$ be a measurable adapted function and $\delta > 0$. By standard density arguments there exists $\tilde{f} \in C([0, T] \times \Omega; \mathbb{S}^d)$ such that

$$E \int_0^T \|\tilde{f}(t, \tilde{W}) - g(t, \tilde{W})\|^2 dt \leq \delta.$$

Let $f(t, x) := \sqrt{\Pi_{\mathbf{D}}(\tilde{f}(t, x)^2)}$. Then $f \in C([0, T] \times \Omega; \sqrt{\mathbf{D}})$ and

$$\|f - g\|^2 \leq \|f^2 - g^2\| \leq \|\tilde{f}^2 - g^2\| \leq (\|\tilde{f}\| + \|g\|)\|\tilde{f} - g\| \leq 2\sqrt{R_{\mathbf{D}}} \|\tilde{f} - g\|$$

(see [1, Theorem X.1.1] for the first inequality). By Jensen's inequality we conclude that $E \int_0^T \|f(t, \tilde{W}) - g(t, \tilde{W})\|^2 dt \leq 2\sqrt{TR_{\mathbf{D}}}\delta$, which, in view of the above steps, completes the proof. \square

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